DETERMINATION OF THE STRESS INTENSITY FACTOR IN ELASTIC PROBLEMS WITH A CRACK

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In contrast to the methods elucidated in [1], a method is proposed in this paper for the solution of problems of plane elasticity theory with a crack: directly in terms of the stresses on the one hand, and on the basis of a finite-difference method on the other. It should be noted that application of the method of finite elements or the method of finite differences is in itself not so essential since proofs exist for the equivalence of these two fundamental numerical methods [2, 3] for a specially selected approximation, while the use of displacements or stresses as the desired functions is more essential. The formulation of the problem in stresses is especially useful when the boundary conditions are also given in stresses, and it is required to investigate the state of stress in the domain given.

Let us consider the plane strain of a rectangular specimen with an edge crack which is propagated in the domain from one of the lateral vertical boundaries in parallel to the two horizontal sides of the rectangle. The crack is symmetric relative to the upper and lower bases; the loads applied to the outer boundaries of the specimen are also symmetric; hence the crack under consideration is a normal discontinuity crack, and the stress state is characterized by one intensity factor K. Because the stresses tend to infinity as the crack nose is approached, it is impossible to use any difference scheme in stresses directly to solve the angular problem under consideration. Hence, it is proposed to use the so-called additive method of extracting singularities in combination with a Rice–Cherepanov integral and the organization of an iterative process using the stress intensity factor to solve the problem.

To apply the additive method of extracting singularities, it is required to have an analytic solution of some auxiliary problem. We may consider as the auxiliary problem, e.g., the problem of tension on a plane with a rectilinear slit along the Ox axis. The tensile loads can be applied both within the domain and on the surface of the slit. The exact (analytic) solution Φ_0 for this singular problem is found by complex-variable function theory methods by reduction to the Riemann–Hilbert conjugate problem and is represented in [4], for instance.

In the initial problem we set the intensity factor K equal to the intensity factor K_0 of the auxiliary problem (this will be the zeroth approximation of the iteration process to be described). A rectangle with an edge crack exactly as in the formulated problem is cut out of a plane M for which the solution Φ_0 is known, and values of the stresses on the boundary of this rectangle, which are known from the solution Φ_0, are added with opposite sign to the boundary conditions of the original problem. The problem obtained is solved numerically by a finite difference method, and a certain stress field φ_0 is found. To find the numerical solution of the problem obtained with bounded continuous boundary conditions, the extended system of equations of plane elasticity theory was integrated

$$\partial L_1/\partial x + (\mathrm{E}/2(1+\nu))S = 0,$$
$$\partial L_2/\partial y + (\mathrm{E}/2(1+\nu))S = 0,\ \partial L_1/\partial y + \partial L_2/\partial x = 0,$$

where

$$L_1 \equiv \partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y;\ L_2 \equiv \partial\sigma_y/\partial y + \partial\tau_{xy}/\partial x;$$
$$S \equiv \frac{1-\nu^2}{\mathrm{E}}\frac{\partial^2\sigma_x}{\partial y^2} - \frac{\nu(1+\nu)}{\mathrm{E}}\frac{\partial^2\sigma_y}{\partial y^2} + \frac{1-\nu^2}{\mathrm{E}}\frac{\partial^2\sigma_y}{\partial x^2} - \frac{\nu(1+\nu)}{\mathrm{E}}\frac{\partial^2\sigma_x}{\partial x^2} - \frac{2(1+\nu)}{\mathrm{E}}\frac{\partial^2\tau_{xy}}{\partial x\partial y}.$$

A finite-difference scheme in stresses of the universal algorithm type was used to approximate this system, in combination with splitting and build-up methods:

$$\left(I - \frac{\tau(2-\nu)}{2}\Lambda_{11}\right)\left(I - \frac{\tau(1-\nu)}{2}\Lambda_{22}\right)\sigma_x^{n+1} = \left(I + \frac{\tau(2-\nu)}{2}\Lambda_{11}\right)\times$$
$$\times\left(I + \frac{\tau(1-\nu)}{2}\Lambda_{22}\right)\sigma_x^n + (1-\nu)\tau\Lambda_{11}\sigma_y^{n+1} - \nu\tau\Lambda_{22}\sigma_y^{n+1},$$
$$\left(I - \frac{\tau(1-\nu)}{2}\Lambda_{11}\right)\left(I - \frac{\tau(2-\nu)}{2}\Lambda_{22}\right)\sigma_y^{n+1} = \left(I + \frac{\tau(1-\nu)}{2}\Lambda_{11}\right)\times$$
$$\times\left(I + \frac{\tau(2-\nu)}{2}\Lambda_{22}\right)\sigma_y^n - \nu\tau\Lambda_{11}\sigma_x^n + (1-\nu)\tau\Lambda_{22}\sigma_x^n,$$
$$\left(I - \frac{\tau}{2}\Lambda_{11}\right)\left(I - \frac{\tau}{2}\Lambda_{22}\right)\tau_{xy}^{n+1} = \left(I + \frac{\tau}{2}\Lambda_{11}\right)\left(I + \frac{\tau}{2}\Lambda_{22}\right)\tau_{xy}^n + \tau\Lambda_{12}(\sigma_x^{n+1} + \sigma_y^{n+1}),$$

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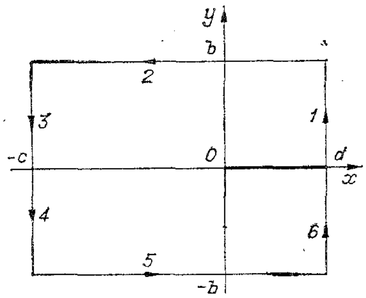

Fig. 1

where $\Lambda_{11}, \Lambda_{22}, \Lambda_{12}$ approximate $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, $\partial^2/(\partial x \partial y)$ in central difference formulas, τ is the iteration parameter, n is the number of the iteration, I is the unit operator, ν is the Poisson ratio, and E is Young's modulus. This scheme was used earlier to solve plane problems of elasticity theory [3]. The stability of this scheme, was investigated by the Fourier method. It should be noted that a similar scheme is presented in [8]. Combining the stress fields φ_0 and Φ_0, we obtain the zeroth approximation to the solution of the formulated problem with a factor K_0 for a singularity at the nose of the crack. We use the J-integral of Rice–Cherepanov [9, 10] to refine this factor and to go over to the next iteration

$$J = \int_{\Gamma} \left(W dy - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x} dS \right),$$

where Γ is an arbitrary contour surrounding the vertex of the crack, W is the strain energy density, and T, u are load and displacement vectors, respectively. For plane strain J and K are connected by the relationship

$$K^2 = [E/(1 - \nu^2)]J.$$

Evaluating the Rice–Cherepanov integral for the stress field $\varphi_0 + \Phi_0$ along a contour sufficiently remote from the crack nose, we obtain the next approximation K_1 of the stress intensity factor. Let us select a real number α such that $K_1 = \alpha K_0$, and let us multiply it by the analytic solution Φ_0 of the auxiliary problem. For the auxiliary problem we obtain the stress field $\Phi_1 = \alpha \Phi_0$ with a factor $K_1 = \alpha K_0$ for the singularity. Now, let us set the intensity factor K_1 in the initial problem for the rectangle with the edge crack (this is the next approximation of the iteration process described). We then repeat the process of extracting the singularities on the basis of the exact auxiliary solution Φ_1. An iteration process in the stress intensity factor has therefore been organized that terminates with the i-th iteration when the condition $|K_{i-1} - K_i| < \varepsilon$, is satisfied, where ϵ is a previously assigned small number characterizing the accuracy of the iteration process constructed.

In contrast to known numerical methods of determining the stress intensity factor [1], this method in particular does not require reducing the mesh size in the neighborhood of the crack nose since infinite values of the stress and large stress gradients are extracted by using the auxiliary exact solution.

The singular solution for a rectilinear crack of a normal discontinuity in an infinite plane can be found by reduction to a boundary-value problem of the theory of functions of a complex variable, called the problem of a linear conjugate of boundary values, or the Riemann–Hilbert conjugate problem [4].

Let us examine two particular cases of the problem mentioned (two auxiliary analytic solutions), that will be used later in solving the original problem.

1. According to [4], the stress components in the plane problem are expressed in terms of two functions of a complex variable $\Phi(z)$ and $\Omega(z)$ by means of the Kolosov–Muskhelishvili formulas:

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z), \qquad \sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z}) \overline{\Phi'(z)}.$$

We obtain in the case of multilateral tension on a plane weakened by a rectilinear crack along the real axis $0 \leqslant x \leqslant 2a$, $y = 0$, when $\sigma_x^{(\infty)} = \sigma_y^{(\infty)} = P$, $\tau_{xy}^{(\infty)} = 0$ at infinity and the crack edges are stress-free [4]

$$\Phi(z) = \Omega(z) = P(z - a)/2\sqrt{z(z - 2a)}.$$

The stress intensity factor is

$$K = P\sqrt{\pi a}.$$

2. Now, let us consider a semiinfinite crack

$$0 \leqslant x \leqslant +\infty, \; y = 0,$$

in an infinite plane. We consider the stresses zero at infinity, and a normal load to be given at the crack edges

TABLE 1

i	α	K
1	0,933109	165,389
2	0,957472	158,355
3	0,973417	154,146
4	0,983562	151,612
5	0,989905	150,081
6	0,993826	149,155
7	0,996234	148,593
8	0,997706	148,252
9	0,998604	148,045
10	0,999151	147,920
11	0,999494	147,843
12	0,999686	147,797
13	0,999809	147,769
14	0,999884	147,752
15	0,999930	147,741

TABLE 2

i	α	K
1	2,000084	70,901
2	1,382279	98,005
3	1,188028	116,433
4	1,102118	128,323
5	1,058233	135,795
6	1,034097	140,426
7	1,020267	143,276
8	1,012152	145,013
9	1,007325	146,075
10	1,004429	146,722
11	1,002683	147,115
12	1,001627	147,355
13	1,000987	147,500
14	1,000600	147,587
15	1,000364	147,642

$$\sigma_y^+ = \sigma_y^- = g(x), \quad \tau_{xy}^+ = \tau_{xy}^- = 0,$$

where

$$g(x) = \begin{cases} 0, & x < a, \quad x > 2a, \\ \dfrac{(x-a)(x-2a)}{\sqrt{x}}, & a \leqslant x \leqslant 2a; \end{cases}$$

The symbols + and – denote the boundary values at the upper and lower edges of the crack. Then

$$\Phi(z) = \Omega(z) = \frac{1}{2\pi i}\left[\frac{2az - 3a^2}{2\sqrt{z}} + \frac{(z-a)(z-2a)}{\sqrt{z}} \ln\frac{2a-z}{a-z}\right].$$

The stress intensity factor is

$$K = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \frac{g(x)}{\sqrt{x}}\,dx = \sqrt{\frac{2}{\pi}}\left(2a^2 \ln 2 - \frac{3a^2}{2}\right).$$

Let us analyze evaluation of the Rice–Cherepanov J-integral written above. It is convenient to select the perimeter of the rectangle displayed in Fig. 1 as the contour Γ. In the Oxy plane the rectangle is the domain [–c, a] X [–b, b]. The contour Γ is traversed counterclockwise. Let us divide it into six segments as is shown in Fig. 1. We represent the J-integral in the form of two integrals

$$J = J_w - J_t,$$

where

$$J_w = \int\limits_\Gamma W dy = \int\limits_0^b W_1 dy + \int\limits_b^0 W_3 dy + \int\limits_0^{-b} W_4 dy + \int\limits_{-b}^0 W_6 dy; \qquad J_t = \int\limits_\Gamma \mathbf{T}\cdot\frac{\partial \mathbf{u}}{\partial x}\,dS.$$

Here the subscript on the W indicates the corresponding segment of the contour Γ. The strain energy density W is determined from the formula

$$W = \frac{1}{2}(\sigma_x\varepsilon_x + 2\tau_{xy}\varepsilon_{xy} + \sigma_y\varepsilon_y),$$

where ϵ_x, ϵ_y, ϵ_{xy} are the strain tensor components.

The integral J_t is evaluated by the formula

$$J_t = \int\limits_{\Gamma_1} T_1 dS + \int\limits_{\Gamma_2} T_2 dS + \ldots + \int\limits_{\Gamma_6} T_6 dS,$$

where

$$T_1 = \sigma_x\varepsilon_x + \tau_{xy}\frac{\partial v}{\partial x}; \quad T_2 = \tau_{xy}\varepsilon_x + \sigma_y\frac{\partial v}{\partial x}; \; T_3 = -\sigma_x\varepsilon_x - \tau_{xy}\frac{\partial v}{\partial x};$$
$$T_4 = -\sigma_x\varepsilon_x - \tau_{xy}\frac{\partial v}{\partial x}; \quad T_5 = -\tau_{xy}\varepsilon_x - \sigma_y\frac{\partial v}{\partial x}; \quad T_6 = \sigma_x\varepsilon_x + \tau_{xy}\frac{\partial v}{\partial x}.$$

To determine the derivative ∂v/∂x with respect to known stresses, and therefore, strains also, we use the relationship

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial \varepsilon_y}{\partial x}, \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = 2\frac{\partial \varepsilon_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y}.$$

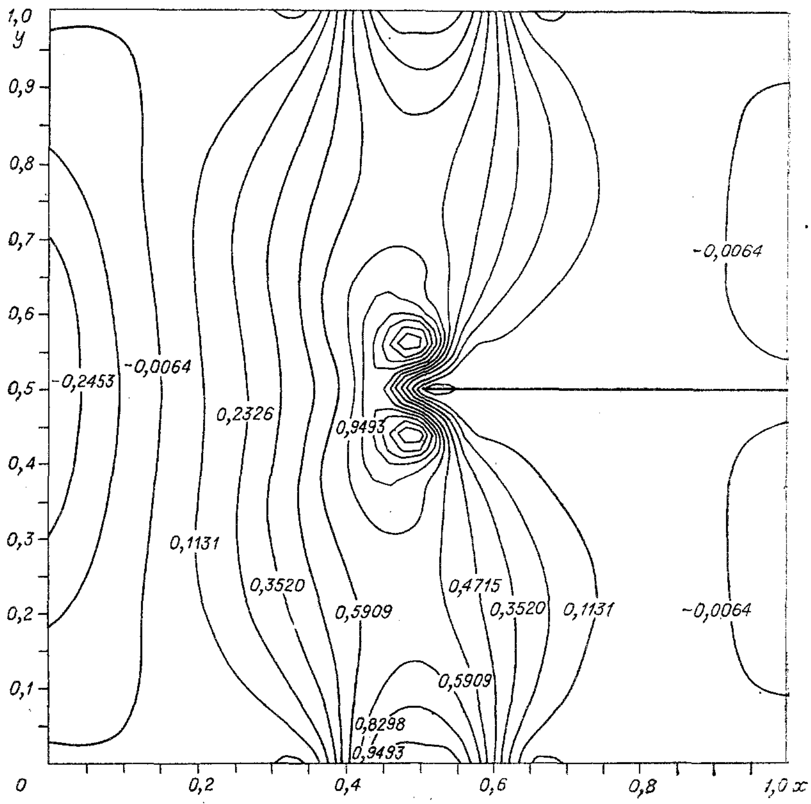

Fig. 2

There results from symmetry conditions that the vertical displacements equal zero on the continuation of the crack in the original problem, therefore, the derivative $\partial v/\partial x$ is also zero. Taking this into account, the relations written down for $\partial v/\partial x$ can be integrated numerically in the whole domain.

An electronic computer solution of the problem is obtained in dimensionless quantities denoted with the upper bar, which are defined as follows

$$\bar{x} = x/l,\ \bar{y} = y/l,\ \bar{\sigma}_x = \sigma_x/E,\ \bar{\sigma}_y = \sigma_y/E,\ \bar{\tau}_{xy} = \tau_{xy}/E,$$

where l is the characteristic linear dimension of the domain. The upper bar will later be omitted over the dimensionless quantities.

Let us select a mesh domain approximating the original domain. Let this be a uniform mesh with step h along x and along y:

$$x_i = hi,\ i = -p,\ -p+1,\ \ldots,\ q-1,\ q,$$
$$y_j = hj,\ j = -r,\ -r+1,\ \ldots,\ r-1,\ r.$$

The following boundary conditions are given for the numerical solution of the problem

$$(\sigma_x)_{-p,j} = (\sigma_x)_{q,j} = 0,\quad j = -r,\ \ldots,\ r,$$
$$(\tau_{xy})_{-p,j} = (\tau_{xy})_{q,j} = 0,\quad j = -r,\ \ldots,\ r,$$
$$(\tau_{xy})_{i,-r} = (\tau_{xy})_{i,r} = 0,\quad i = -p,\ \ldots,\ q,$$
$$(\sigma_y)_{i,-r} = (\sigma_y)_{i,r} = \begin{cases} 0, & i = -p,\ \ldots,\ -3, 3,\ \ldots,\ q, \\ 1, & i = -1, 0, 1, \\ 1/2, & i = -2, 2. \end{cases}$$

The domain dimensions were taken as p = q = r = 20, h = 0.025. Alloyed steel with the elastic constants $\nu = 0.2875$, E = 20,998.98 kg/mm² was taken as specimen material.

The initial problem for a rectangle with an edge crack was investigated in detail by means of the algorithm proposed. The following versions of this problem were hence examined.

1. The first analytic solution with a load at infinity P = 100/E, α = 1 was taken as auxiliary problem. The results are presented in Table 1, where i is the number of the iteration, K is the stress intensity factor, and the parameter α is the ratio of two successive values of K.

2. The first analytic solution with a load P = 20/E at infinity was taken as the auxiliary problem. In this case the successive values of the intensity factor converged to the desired value from below (Table 2).

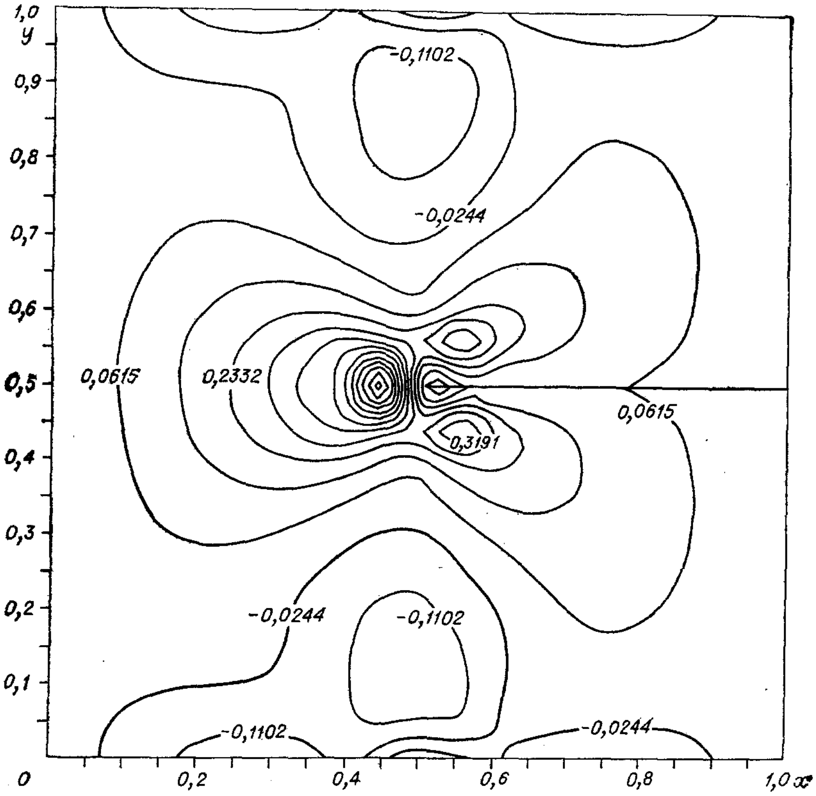
1,0
y
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0,2
0,4
0,6
0,8
1,0 x
-0,1102
-0,0244
0,0615
0,2332
0,0615
0,3191
-0,0244
-0,1102
-0,1102
-0,0244

Fig. 3

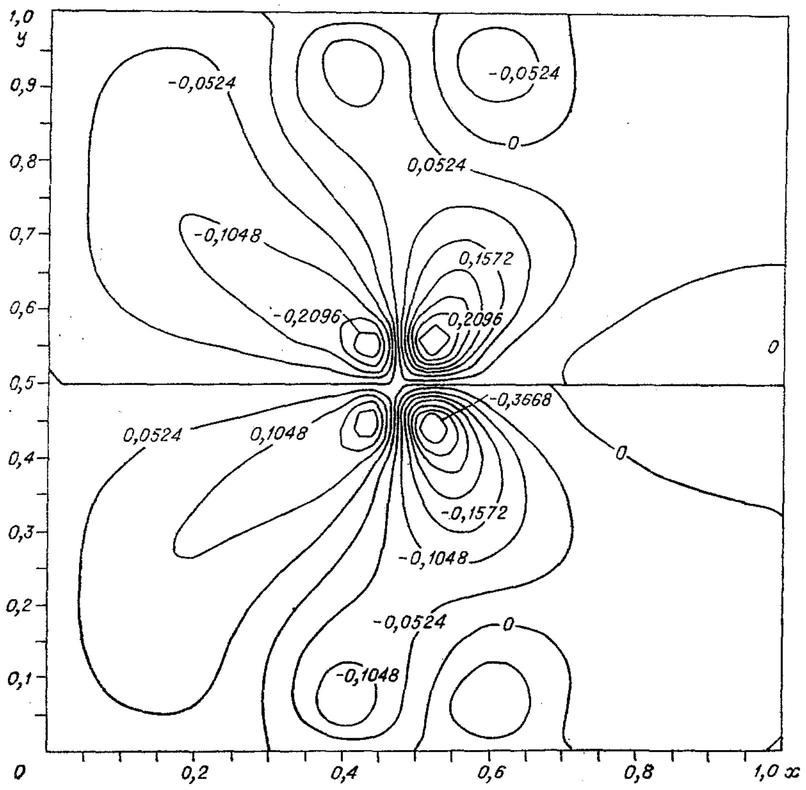
1,0
y
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0,2
0,4
0,6
0,8
1,0 x
-0,0524
-0,0524
0
0,0524
-0,1048
0,1572
-0,2096
0,2096
0
-0,3668
0,0524
0,1048
0
-0,1572
-0,1048
-0,0524
0
-0,1048

Fig. 4

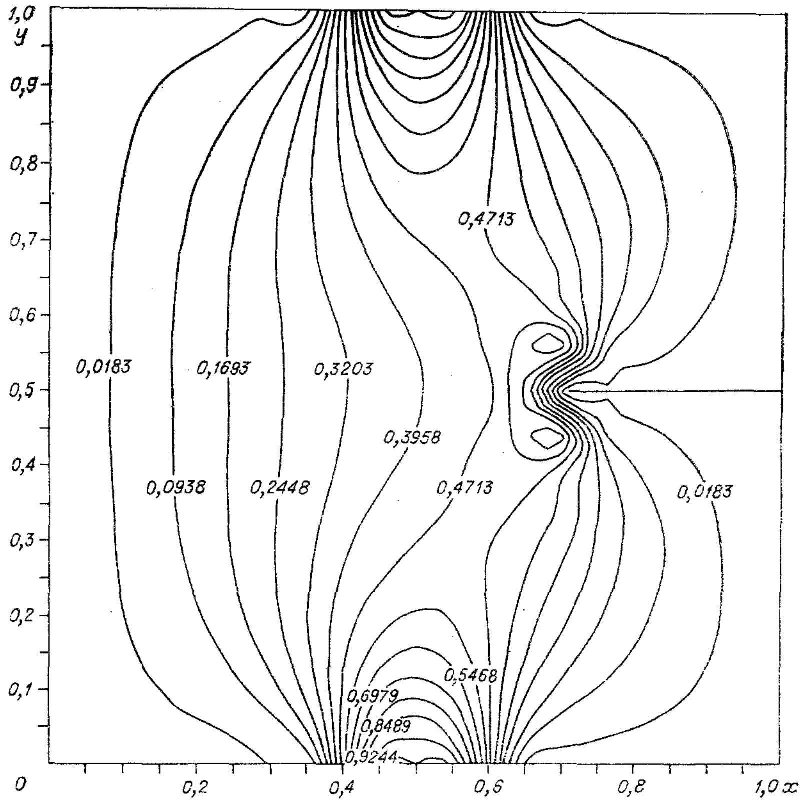

Fig. 5

3. The second analytic solution, where $a = 1$, was taken as auxiliary problem. The value of the stress intensity factor was 156.579 at the 15-th iteration, which is quite close to the intensity factor of the first problem considered.

The crack length was half the length of a side of the square in all the problems listed above.

4. A problem was examined when the crack length was 1/3 the length of a side of the square: p = 26, q = 14, r = 20, and the boundary conditions in the initial problem were not changed here. The value of the intensity factor at the 15-th iteration was K = 58.385.

5. To investigate the question of accuracy of the method, the following test problem was solved: Values of the stresses on the outline of a square were taken from the first analytic solution as boundary conditions. The intensity factor is known $K = P\sqrt{\pi a} = 177.245$ for P = 100, $a = 1$. The second exact (analytic) solution was taken as auxiliary solution. The value of the stress intensity factor at the fifteenth iteration 186.921 differed by 5.64% from the actual value.

Displayed in Figs. 2-4 are isobars of the functions σ_y/E, σ_x/E, τ_{xy}/E for a crack length of 1/2 the side of the square. A graph of the function σ_y/E for a crack length of 1/3 the side of the square is presented in Fig. 5. The stresses σ_y, σ_x are symmetric relative to the Ox axis, and τ_{xy} is anti-symmetric. Stress concentration is observed at the crack nose in all the graphs. All the stresses tend to infinity as the crack nose is approached. Because of technical difficulties in constructing the graphs in Figs. 2-5, the stresses at the crack nose were assumed limited although the method applied yields infinite values of the stress at the nose. The function σ_y in Fig. 5 is positive in the whole domain, reaches the greatest value at the crack nose, and there is also a local maximum in the neighborhood of the external load application. The stresses σ_y in Fig. 2 decrease in going from the crack nose to the left side boundary, and become negative in a domain adjoining the upper and lower bases, and positive in the central domain. It is seen that the stresses σ_y exceed, in absolute value, the stresses σ_x in the whole domain at corresponding points.

We note that the graphs given here were constructed using the graph-drawing program BENSON OFF B-220 – a special implementation was designed for the BÉSM-6 computer in the Computing Center of the Siberian Branch of the Academy of Sciences of the USSR; the program used a mathematical method of graph construction.

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EFFECT OF ANGLE OF ATTACK OF A METAL SURFACE ELEMENT ON THE ENERGY ACCOMODATION COEFFICIENT OF NITROGEN IONS

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In computing aerodynamic characteristics and heat transfer for vehicles in free-molecular flow it is important to know the energy accommodation coefficient for incident stream particles and its dependence on the orientation of a surface element relative to the flow velocity vector. The literature does not have the required volume of information on the accomodation coefficient as a function of surface orientation for the particle energy range of practical interest, ~1-100 eV. The present paper reports the dependence of the accommodation coefficient α_i for nitrogen ions on the angle of attack of metal targets with atomic weight in the range 27 to 197, measured in a high-speed rarefied plasma flow ($u_\infty \simeq 10$ km/sec).

The experimental investigations were conducted in a gasdynamic plasma facility in a flow of partially ionized gas, generated by an accelerator in which the working substance was ionized by an electron beam. The accelerated ion flux, of intensity $j_\infty \simeq 10^{17}$ ion/cm²•sec was directed into the working chamber, where the residual gas pressure was $\sim 7\cdot 10^{-7}-1\cdot 10^{-6}$ torr. The measurements were done at a working chamber pressure of $\sim(0.87\text{-}1.6)\cdot 10^{-5}$ torr.

To measure the accommodation coefficient of the nitrogen ions we used a planar hot wire anemometer probe, in the form of a disk of thickness $\delta \simeq 0.12$ mm with a working surface diameter of 3.5 mm, and with current leads and a thermocouple attached to its back face. The lateral surface of the sensor, the thermocouple, and the current leads were insulated from contact with the ceramic plasma tube.

A rake of sensors with working surfaces made of different materials was set up in the high-speed stream of rarefied plasma. The volt–ampere characteristics $\lg I_e = f(V)$ had a clearly pronounced straight-line section. Thus, we could determine the electron temperature $T_e \simeq 3.5\text{-}4.7$ eV ($W_e = 2kT_e$) by the usual method [1]. The plasma potential φ_0 was determined by the second derivative method, and also from the electron part of the probe characteristic. This gave high accuracy in measuring the stream ion energy W_i. The values of W_i obtained agree satisfactorily with values found by use of a multielectrode analyzer probe, and also with values calculated on the assumption that the accelerating potential is the difference between the source anode and the local plasma potential φ_0. The scatter in the values of W_i obtained does not exceed ± 4.5%. To check the local values of the flow operating parameters and the orientation of the sensors relative to the flow vector u_∞ we used a slender cylindrical probe made of molybdenum wire of diameter 0.09 and length 4.0 mm. The peak ion current measured by this probe, when rotated about horizontal and vertical axes, corresponds to the probe orientation in the flow [2], and allows an estimate to be made of the degree of nonisothermality of the flow $T_i/T_e \simeq 0.13$.

The ion energy accommodation coefficient α_i was determined, using the technique of [3], from the relation

$$\frac{\dot{I}_i^A}{e}\{\xi+\alpha_i(W_i+e|V^A|)-\gamma_i\varkappa\}+\frac{\dot{I}_e^A}{e}(W_e+\varkappa)=\frac{\dot{I}_e^B}{e}(W_e+\varkappa+e|V^B|), \quad (1)$$

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